Calabi–Yau Varieties
Arithmetic, Geometry and Physics

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2016 UNAM

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Modularity/Automorphy Questions

A: Arithmetic modularity/automorphy. This is concerned with Calabi–Yau varieties defined over $\mathbb{Q}$ (or number fields). The main questions are the modularity/automorphy of the cohomological L-series of the ℓ-adic Galois representations associated to these varieties in the framework of Langlands Philosophy.

G: Geometric modularity. This is concerned with families of Calabi–Yau manifolds. The main questions are the modular/automorphic properties of various invariants associated to these varieties, e.g., mirror maps, Gromov–Witten invariants, Donaldson–Thomas invariants, holomorphic anomaly equations, etc. arising in string theory. Quasi-modular forms, Jacobi forms, Siegel modular forms, and more general modular-like forms show their appearances in this landscpae.

Langlands Philosophy for Arithmetic Modularity

(Motivic) L-functions of algebraic varieties over $\mathbb{Q}$ (or number fields) are automorphic L-functions

I will try to give some examples in support of this philosophy when varieties are Calabi–Yau varieties defined over $\mathbb{Q}$.

In this talk, we will consider Calabi–Yau varieties defined over $\mathbb{Q}$ of dimension at most 3.

Calabi–Yau Varieties

Definition: A smooth projective variety $X/\mathbb{C}$ of dimension d is said to be *Calabi–Yau* if

(CY1) $H^i(X, \mathcal{O}_X) = 0 \quad$ for every i, $0 < i < d$, and

(CY2) The canonical bundle $\mathcal{K}_X$ is trivial, i.e, $\mathcal{K}_X \simeq \mathcal{O}_X$.

Now introduce Hodge numbers:

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega^i_X) \quad \text{for } 0 \leq i, j \leq d$$

Then

$$h^{i,j}(X) = h^{j,i}(X) \quad \text{by complex conjugation}$$

and

$$h^{i,j}(X) = h^{d-i,d-j}(X) \quad \text{by the Serre duality.}$$

Remark In terms of Hodge numbers, $X/\mathbb{C}$ is Calabi–Yau if

(CY1)$\Leftrightarrow$ $h^{i,0}(X) = 0 \quad$ for every $i, 0 < i < d$, and

(CY2)$\Rightarrow$ $h^{d,0}(X) = h^{0,d}(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^d) = \dim_{\mathbb{C}} H^0(X, \mathcal{K}_X)$
$= \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1.$

The number $h^{0,d}(X)$ is the *geometric genus* $p_g(X)$ of X.

Numerical characters

- The Betti numbers $B_k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$.

$$B_k(X) = \sum_{i+j=k} h^{i,j}(X).$$

- The Euler characteristic $E(X) := \sum_{k=0}^{2d} (-1)^k B_k(X)$.

Hodge diamonds

The Hodge numbers of Calabi–Yau varieties are concocted to form the Hodge diamond.

$d = 1$: Elliptic curves

$$h^{1,0} = h^{0,1} = 1$$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \qquad \begin{array}{r} B_0 = 1 \\ B_1 = 2 \\ B_2 = 1 \\ E = 0 \end{array}$$

Dimension one Calabi–Yau varieties are elliptic curves. Elliptic curves over $\mathbb{Q}$ are defined by $y^2 = x^3 + ax + b$ with $4a^3 + 27b^2 \neq 0$.

$d = 2$: K3 surfaces

$$h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = 1$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{c} B_0 = 1 \\ B_1 = 0 \\ B_2 = 22 \\ B_3 = 0 \\ B_4 = 1 \\ E = 24 \end{array}$$

By the Max Noether's formula, we have

$$\chi(\mathcal{O}_X) = (c_1^2 + c_2)/12$$

where c_1, c_2 are the first and the second Chern numbers, and

$$c_1 = 0,\ c_2 = E = 24.$$

Now

$$\chi(\mathcal{O}_X) = h^{0,0} - h^{1,0} + h^{2,0} = 1 - 0 + 1 = 2$$

from which we can derive that $h^{1,1} = 20$.

Examples: (1) Any quartic surface in $\mathbb{P}^3$. A typical example is the Fermat quartic:

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0 \subset \mathbb{P}^3,$$

or its one-parameter deformation:

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4\lambda X_0 X_1 X_2 X_3 = 0 \subset \mathbb{P}^3 \times \mathbb{P}^1.$$

(This is bacause, $\mathcal{K}_X \simeq \mathcal{O}_X(d-N-1) \simeq \mathcal{O}_X$ implies that $d = N+1$. If $N = 3$, then $d = 4$.)

(2) Double sextic surface, e.g., $w^2 = f_6(x, y, z)$.

(3) Elliptic K3 surfaces, e.g., $Y^2 Z = X^3 + A(t) X Z^2 + B(t) Z^3$ with $4A^3(t) + 27B^2(t) \neq 0$.

(4) Complete intersections.

(5) Toric constructions, reflexive polytopes.

$d = 3$: Calabi–Yau threefolds

$$h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = 0, \quad h^{3,0} = h^{0,3} = 1, h^{1,1} > 0$$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

$$\begin{aligned}
B_0 &= 1 \\
B_1 &= 0 \\
B_2 &= h^{1,1} \\
B_3 &= 2(1 + h^{2,1}) \\
B_4 &= h^{2,2} \\
B_5 &= 0 \\
B_6 &= 1 \\
E &= 2(h^{1,1} - h^{2,1})
\end{aligned}$$

Problem: $|E(X)| < \infty$?

Examples: (1) Quintic threefolds in $\mathbb{P}^4$. A typical example is is the Fermat quintic:

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0 \subset \mathbb{P}^4,$$

or its one-parameter deformation:

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - 5\lambda X_0 X_1 X_2 X_3 X_4 = 0 \subset \mathbb{P}^4 \times \mathbb{P}^1.$$

More generally, hypersurface Calabi–Yau threefolds in projective/weighted projective spaces.

(2) Double octics, fiber products, Calabi–Yau threefolds of product type.

(3) Complete intersection threefolds.

(4) Toric Calabi–Yau threefolds ($\sim$ 600 million).

The largest possible Hodge numbers, or equivalently the Euler characteristic of a Calabi–Yau threefold is not known, but some known examples (~ 600 million) have $h^{1,1}$(or $h^{2,1}$)~ 500.

An implication in string theory is that string theory may have as many as 10^{500} vacua that can be described with various choices of branes and fluxes on homology cycles of a CY. Thus string theory has a vast number of different vacua, and only one should describe dynamics of our real world.

(Topological) Mirror Symmetry Conjecture

Given a Calabi–Yau threefold X with Hodge numbers $(h^{1,1}(X), h^{2,1}(X))$, there exists a mirror family of Calabi–Yau threefolds $\hat{X}$ with Hodge numbers $(h^{1,1}(\hat{X}), h^{2,1}(\hat{X}))$ such that

$$h^{1,1}(\hat{X}) = h^{2,1}(X),\ h^{2,1}(\hat{X}) = h^{1,1}(X)$$

so

$$E(\hat{X}) = -E(X).$$

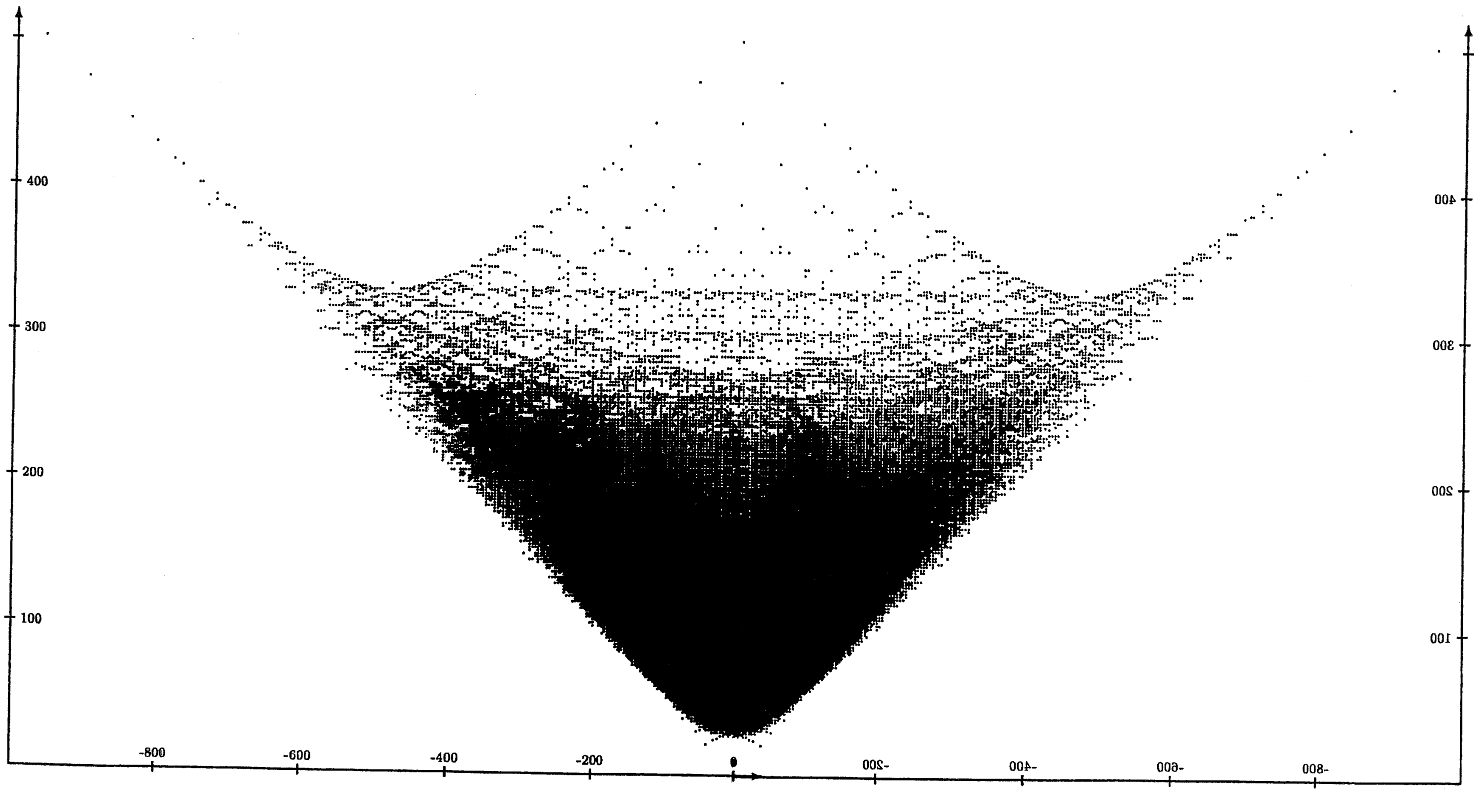

Fig. 1: $h_{11} + h_{12}$ vs. Euler number $\chi = 2(h_{11} - h_{12})$ for all pairs (h_{11}, h_{12}) with $h_{11} \le h_{12}$.

The L-series

We will consider Calabi–Yau varieties defined over $\mathbb{Q}$, say, by hypersurfaces or by complete intersections. We say that $X/\mathbb{Q}$ is Calabi–Yau if $X \otimes_{\mathbb{Q}} \mathbb{C}$ is Calabi–Yau. Let $X/\mathbb{Q}$ be a Calabi–Yau with a defining equation with coefficients in $\mathbb{Z}[1/m]$ for some $m \in \mathbb{N}$. Let p be a prime $(p, m) = 1$, let $X_p := X \bmod p$ be the reduction of X modulo p. We say that p is *good* if X_p is smooth over $\overline{\mathbb{F}}_p$, otherwise *bad*.

Let $\#X(\mathbb{F}_{p^k})$ be the number of rational points on X_p over $\mathbb{F}_{p^k}$. The *local (congruent) zeta function* of X_p is defined by taking the formal sum

$$Z_p(X, T) := \exp\left(\sum_{k=1}^{\infty} \frac{\#X(\mathbb{F}_{p^k})}{k} T^k\right) \in \mathbb{Q}[[T]]$$

where T is an indeterminate.

There were vast series of conjectures about $Z_p(X,T)$, known as the Weil conjectures, proved finally by Deligne.

Let ℓ be a prime $\neq p$. The Frobenius morphism Fr_p $(x \mapsto x^p)$ on X_p induces an endomorphism Fr_p^* on the étale cohomology groups $H^i_{et}(\overline{X}_p, \mathbb{Q}_\ell)$ for each i, $0 \leq i \leq 2d$. Grothendieck specialization theorem gives an isomorphism $H^i_{et}(\overline{X}_p, \mathbb{Q}_\ell) \simeq H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$, where $\overline{X} = X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. By the comparison theorem, $H^i_{et}(\overline{X}, \mathbb{Q}_\ell) \simeq H^i(X \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C})$ so that $\dim_{\mathbb{Q}_\ell} H^i_{et}(\overline{X}, \mathbb{Q}_\ell) = B_i(X)$ (the i-th Betti number). There is the Poincaré duality: $H^i(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}_{et}(\overline{X}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ is a perfect pairing for every i, $0 \leq i \leq 2d$.

Let

$$P_p^i(T) := \det(1 - \mathrm{Fr}_p^*\, T \mid H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

be the characteristic polynomial of Fr_p^*.

Weil's Conjectures (Theorem)

- $P_p^i(T) \in 1 + T\mathbb{Z}[T]$.
- $P_p^i(T)$ does not depend on the choice of ℓ.
- $\deg P_p^i(T) = B_i(X)$ for every $i, 0 \leq i \leq 2d$.
- $P_p^{2d-i}(T) = \pm P_p^i(p^{d-i}T)$ for every i, $0 \leq i \leq d$.
- If we write $P_p^i(T) = \prod_{k=1}^{B_i}(1 - \alpha_k\, T) \in \overline{\mathbb{Q}}[T]$, then α_k is an algebraic integer with $|\alpha_k| = p^{i/2}$ (The Riemann Hypothesis).
- $Z_p(X,T)$ is a rational function:

$$Z_p(X,T) = \frac{\prod_{i=1}^d P_p^{2i-1}(X,T)}{\prod_{i=0}^d P_p^{2i}(X,T)}.$$

Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. There is a compatible system of ℓ-adic Galois representations

$$\rho^i_{X,\ell} : G_{\mathbb{Q}} \to GL(H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

sending the (geometric) Frobenius $\mathrm{Fr}{*}_p^{-1}$ to $\rho^i(\mathrm{Fr}_p^{*-1})$ which has the same action as the Fr_p^* on $H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$.

Definition: The i-th (cohomological) L-series (or L-function) of $X/\mathbb{Q}$ is defined by

$$L_i(X, s) := L(H^i_{et}(\overline{X}, \mathbb{Q}_\ell), s)$$

$$:= (*) \prod_{p \neq \ell : \text{good}} P^i_p(p^{-s})^{-1} \times (\text{factor corresponding to } \ell = p)$$

where the product is taken over all good primes different from ℓ and $(*)$ corresponds to factors of bad primes. For $\ell = p$ we use p-adic cohomology groups.

The most significant L-series is the L-series $L_d(X, s) =: L(X, s)$.

The vector space $H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$ may decompose into a direct sum of subspaces, and we can define the *motivic* L-series corresponding to these subspaces.

Locally for each good prime, the characteristic polynomial $P^i_p(T)$ can be determined by geometric information and by counting the number of rational points on $\mathbb{F}_p$ by invoking the Lefschetz fixed point formula.

$$\#X(\mathbb{F}_p) = \sum_{k=0}^{2d} (-1)^k \mathrm{trace}(\mathrm{Fr}_p^* \,|\, H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

Automorphy/Modularity Question

Are there global functions that determine the L-series $L_i(X, s)$?

More concretely, are there automorphic (modular) forms that determine $L_i(X, s)$?

Modularity Results since 1994

• **Dim 1**: Every elliptic curve E over $\mathbb{Q}$ is modular. There is a modular form f of weight 2 on some $\Gamma_0(N)$ such that $L(E,s) = L(f,s)$.

• **Dim 2**: Every singular K3 surface S over $\mathbb{Q}$ is modular. There is a modular form f of weight 3 on some $\Gamma_0(N)$ with a mod N Dirichelt character χ such that $L(T(S) \otimes \mathbb{Q}_\ell, s) = L(f, \chi, s)$.

• **Dim 3**: Every rigid Calabi–Yau threefold X over $\mathbb{Q}$ is modular. There is a modular form f of weight 4 on some $\Gamma_0(N)$ such that $L(X,s) = L(f,s)$.

Modular forms

• Modular groups. Let

$$SL_2(\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}.$$

Put $\Gamma = PSL_2(\mathbf{Z}) = SL_2(\mathbf{Z})/\pm I_2$. Then

$$= \left\langle\, S, T \mid S^2 = (TS)^3 = I_2 \,\right\rangle$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Γ acts on the upper-half complex plane $\mathfrak{H} = \{\, z \in \mathbf{C} \mid \mathrm{Im}(z) > 0 \,\}$

by the linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

Define $\mathfrak{H}^* = \mathfrak{H} \cup \mathbf{P}^1(\mathbb{Q})$, where $\mathbf{P}^1(\mathbb{Q})$ is the set of fixed points (called cusps) of this action. ($(x:1) \in \mathbf{P}^1(\mathbb{Q})$ is identified with $x \in \mathbb{Q}$; $(x:0)$ with $i\infty$.)

Now let $N \geq 1$ be an integer. Let

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, c \equiv 0 \pmod{N} \}.$$

$\Gamma_0(N)$ is a congruence subgroup of Γ.

Let k be a non-negative integer.

Definition: A *modular form of weight k and level N* on $\Gamma_0(N)$ is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

(MF1)

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

(MF2) f is holomorphic at all cusps.

Since $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, f has a Fourier expansion at $i\infty$:

$$f(q) = \sum_n a_f(n) q^n \text{ with } q = e^{2\pi i z}.$$

The cusp $i\infty$ corresponds to $q = 0$, and f is holomorphic at $i\infty$ if and only if $a_f(n) = 0$ for all $n < 0$; f vanishes at $i\infty$ if $a_f(0) = 0$. A cups form is a modular form which vanishes at all cusps.

We can also define modular forms with a mod N character χ:

$$f(\frac{az+b}{cz+d}) = \chi(d)(cz+d)^k f(z)$$

and

$$f(q) = \sum_n \chi(n) a_f(n) q^n.$$

Result: $d = 1$

Theorem (Wiles et al. 1994,..., 2001.): *Every elliptic curve E over $\mathbb{Q}$ is modular. That is, there is a cusp form f of weight $2 = 1 + 1$ on $\Gamma_0(N)$ such that*

$$L(E, s) = (*) \prod_{p:\mathit{good}} P_p^1(p^{-s})^{-1} = L(f, s)$$

Here N is the conductor of E and $\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \} \subset PSL(2, \mathbb{Z})$.

There is a compatible system of 2-dimensional ℓ-adic Galois representations associated to E, and Wiles et al. established its modularity.

• $L(E, s)$ for an elliptic curve E over $\mathbb{Q}$.

For a good prime p, we have

$$\#E(\mathbf{F}_p) = 1 - t_1(p) + p$$

where $t_1(p) = \mathrm{trace}(Fr_p^*|H^1_{et}(\overline{E}, \mathbb{Q}_\ell))$ with $|t_1(p)| \leq 2p^{1/2}$. Then

$$P^1_p(T) = 1 - t_1(p)T + pT^2$$

and $L(E, s)$ is given by

$$L(E, s) = (*) \prod_{p:good} P_1(p^{-s})^{-1} = \sum_{n \geq 1} \frac{a(n)}{n^s}.$$

On the other hand, write the Fourier expansionof a modular (cusp) form f of weight 2 and level N as

$$f(q) = \sum_{n \geq 1} \frac{a_f(n)}{n^s}$$

with $a_f(1) = 1$. Then

$$a(n) = a_f(n) \quad \forall n.$$

Remark: However, there is no conceptial understanding why E is modular. Perhaps, this may be explained by some “index” theorem?

Some Results $d = 3$

Definition: A Calabi–Yau threefold X over $\mathbb{Q}$ is said to be *rigid* if $h^{2,1}(X) = 0$ (so that $B_3(X) = 2$). Thus, the Hodge diamond of any rigid Calabi–Yau threefold is given by

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{c}
B_0 = 1 \\
B_1 = 0 \\
B_2 = h^{1,1} \\
B_3 = 2 \\
B_4 = h^{2,2} \\
B_5 = 0 \\
B_6 = 1 \\
E = 2h^{1,1}
\end{array}
$$

Theorem (Gouvêa–Yui/Dieulefait) :*Every rigid Calabi–Yau threefold X defined over $\mathbb{Q}$ is modular. That is, there is a cusp form f of weight $4 = 3 + 1$ on $\Gamma_0(N)$ such that*

$$L(X, s) = L_3(X, s) = L(f, s)$$

Here N is divisible only by bad primes.

There is a compatible system of 2-dimensional ℓ-adic Galois representations associated to X. Proof relies on the validity of Serre's conjecture on the residual 2-dimensional Galois representations, proved by Khare–Wintenberger and Kisin.

Let A_3 be the root lattice. Associated to A_3, Verrill constructed a rigid Calabi–Yau threefold X over $\mathbb{Q}$. It is defined as a smooth resolution of the hypersurface

$$(X_1+X_2+X_3+X_4)(X_1^{-1}+X_2^{-1}+X_3^{-1}+X_4^{-1})-(t-1)^2/t-4=0$$

in $\mathbf{P}^3 \times \mathbf{P}^1$. Then

$$h^{3,0}=1, h^{1,0}=h^{2,0}=0,\ h^{2,1}=0,\ h^{1,1}=50$$

So X is a rigid Calabi–Yau threefold over $\mathbb{Q}$ with $E(X)=100$. Thus, X is modular, that is,

$$L(X,s)=L(f,s)$$

where

$$f(q)=[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)]^2$$

is a weight 4 modular (cusp) form for $\Gamma_0(6)$, where $\eta(q)$ is the eta-function.

Masahiko Saito and Yui (2001) constructed a rigid Calabi–Yau threefold Y defined over $\mathbb{Q}$. Let S be the rational elliptic surface defined by the hypersurface

$$(x+y+z)(xy+yz+zx) = (s+1)xyz \subset \mathbf{P}^2 \times \mathbf{P}^1$$

associated to $\Gamma_1(6)$. Put $Y_0 = S \times_{\mathbf{P}^1} S$ be the self-fiber product of S and Y be a crepant resolution of Y_0. Then Y is a rigid Calabi–Yau threefold with $h^{1,1} = 50$.

There is an explicit birational transformation defined over $\mathbb{Q}$ from Y to X. Therefore,

$$L(Y,S) = L(X,s) = L(f,s).$$

(Note that $\Gamma_0(6)$ and $\Gamma_1(6)$ have the same projectivation.)

• $L(X,s)$ for a rigid Calabi–Yau threefold X over $\mathbb{Q}$.

For a good prime p, we have

$$\#X(\mathbf{F}_p)=\sum_{i=0}^{6}(-1)^i t_i(p)$$

where we put $t_i(p)=\mathrm{trace}(Fr_p^*|H^i_{et}(\overline{X},\mathbb{Q}_\ell))$. Then

$$\#X(\mathbf{F}_p)=t_0(p)-t_1(p)+t_2(p)-t_3(p)+t_4(p)-t_5(p)+t_6(p)$$

$$=1+t_2(p)-t_3(p)+t_4(p)+p^3$$

$$=1+p^3+(1+p)t_2(p)-t_3(p)$$

So

$$t_3(p)=1+p^3+(1+p)t_2(p)-\#X(\mathbf{F}_p)$$

with

$$|t_2(p)|\le ph^{1,1},\quad \text{and}\quad |t_3(p)|\le 2p^{3/2}.$$

Then

$$P_p^3(T) = 1 - t_3(p)T + p^3T^2$$

and the $L(X,s)$ is given by

$$L(X,s) = (*) \prod_{p:\text{good}} P_3(p^{-s})^{-1} = \sum + n\frac{a(n)}{n^{-s}}$$

On the other hand, we have the Fourier expansion of a modular form f of weight 4 on $\Gamma_0(N)$ for some $N \in \mathbb{N}$:

$$f(q) = \sum_{n\geq 1} \frac{a_f(n)}{n^s}.$$

Then we have

$$a(n) = a_f(n) \quad \forall n.$$

Some Results: $d = 2$

Let X be a K3 surface defined over $\mathbb{Q}$. Then $H^2(X, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 22. There is the intersection pairing on X, which gives rise to a quadratic form on $H^2(X, \mathbb{Z})$, and $H^2(X, \mathbb{Z})$ captures a structure of a lattice of rank 22 with the following properties: unimodular, even, indefinite and with signature $(3, 19)$. Then there is an isometry

$$H^2(X, \mathbb{Z}) \to U^3 \oplus (-E_8)^2$$

where U is the hyperbolic lattice of rank 2 and $-E_8$ is the unique negative definite unimodular form of rank 8. The latter is called the *K3 lattice.*

Let $NS(X)$ denote the Néron–Severi group of X generated by algebraic cycles. It is a free finitely generated abelian group, and $NS(X) = H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$ so that the rank of $NS(X)$ (called the Picard number of X and denoted by $\rho(X)$) is bounded by 20. Let $T(X) = NS(X)^{\perp}$ be the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. It has the $\mathbb{Z}$-rank $22 - \rho(X)$, and is called the group of transcendental cycles on X. We have the decomposition

$$H^2(X, \mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = (NS(X) \otimes \mathbb{Q}_\ell) \oplus (T(X) \otimes \mathbb{Q}_\ell)$$

and we have the decomposition of the L-series:

$$L_2(X, s) = L(NS(X) \otimes \mathbb{Q}_\ell, s) L(T(X) \otimes \mathbb{Q}_\ell, s).$$

The Tate conjecture is valid for any K3 surface over $\mathbb{Q}$, which asserts that

$$H^2_{et}(\overline{X}, \mathbb{Q}_\ell)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = NS(X)_{\mathbb{Q}}.$$

Thus the L-series $L(NS(X) \otimes \mathbb{Q}_\ell, s)$ is expressed in terms of $\zeta(s-1)^{\rho(X)}$ if all algebraic cycles are defined over $\mathbb{Q}$ where $\zeta(s)$ denotes the Riemann zeta-function. The other extreme is when all algebraic cycles are defined over some algebraic number field $\mathbb{L}$, then $L(NS(X) \otimes \mathbb{Q}_\ell, s) = \zeta_{\mathbb{L}}(s-1)^{\rho(X)}$ where $\zeta_{\mathbb{L}}(s)$ is the Dedekind zeta-function of $\mathbb{L}$. However, these extreme situations occur very rarely. In general, some algebraic cycles may be defined over $\mathbb{Q}$, but others are not, in which case, Artin L-function should come into the picture.

Therefore, for K3 surfaces, we will address the modularity/automorphy of the motivic L-series, namely, that of

$$L(T(X) \otimes \mathbb{Q}_\ell, s).$$

Definition: A K3 surface X over $\mathbb{Q}$ is *singular* (or *extremal*) if $\rho(X) = 20$ (so that $\mathbb{Z}$-rank of $T(X) = 2$).

Theorem (Livné): *Every singular K3 surface X over $\mathbb{Q}$ is motivically modular. That is, there is a cusp form f of weight $3 = 2 + 1$ on $\Gamma_0(N)$ with a character χ such that*

$$L(T(X) \otimes \mathbb{Q}_\ell, s) = L(f, \chi, s).$$

Here χ is a mod N Dirichlet character associated to an imaginary quadratic field over $\mathbb{Q}$.

There is a compatible system of 2-dimensional ℓ-adic Galois representations associated to $T(X)$, and Livné established the modularity of such representations.

Mirror symmetry conjecture for K3 surfaces

This version of mirror symmetry for K3 surfaces is due to Dolgachev (based on Arnold's strange duality).

Let X be a K3 surface with Picard number $\rho(X)$. Then there exists a mirror K3 surface $\hat{X}$ with Picard number $\rho(\hat{X})$ such that

$$T(X) = U \oplus NS(\hat{X})$$

in terms of rank,

$$22 - \rho(X) = 2 + \rho(\hat{X}) \Leftrightarrow \rho(X) + \rho(\hat{X}) = 20.$$